# Collective mode dispersion in a two-dimensional quantum dipole systems

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# Outline

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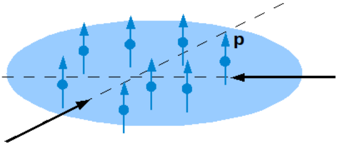
# Systems with dipole interaction

**Experimental realizations*:**

- Atoms with large permanent magnetic moment, e.g. $^{52}$Cr
- Polar molecules
- Indirect excitons in quantum wells




* Greismaier, Lahaye, Butov, Snoke, Timofeev, et al.

**Spatial confinement:**

- Magneto-optical traps; Quantum Stark confinement (excitons)

**Hamiltonian:** 2D homogeneous systems of $N$ dipolar bosonic particles

$$\hat{H} = -\sum_{i=1}^{N} \frac{\hbar^2 \nabla_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} \frac{p^2}{|\mathbf{r}_i - \mathbf{r}_j|^3}.$$

# 2D dipoles: Classical and quantum coupling parameters

**Coupling parameter**, $\Gamma(n,\beta) = \langle E_{\text{pot}}\rangle \,/\, \langle E_{\text{kin}}\rangle$, depends on:

- Particle density, $n = N/V = 1/\bar{r}^2 = 1/\pi\bar{a}^2$; Temperature, $\beta = 1/k_B T$
- Average potential energy, $\langle E_{\text{pot}}\rangle\,(n,\beta)$
- Average kinetic energy, $\langle E_{\text{kin}}\rangle\,(n,\beta)$

**Classical regime**

$\langle E_{\text{pot}}\rangle \sim p^2/\bar{a}^3, \quad \langle E_{\text{kin}}\rangle = 1/\beta$

Coupling: $\Gamma_D = \beta p^2/\bar{a}^3$

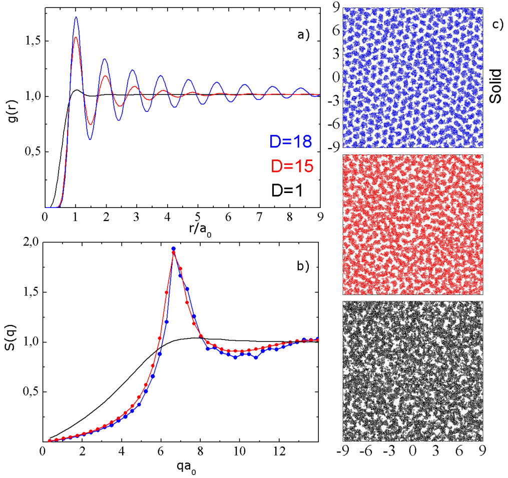
**Quantum regime**

$\langle E_{\text{pot}}\rangle \sim p^2/\bar{r}^3, \quad \langle E_{\text{kin}}\rangle \sim \hbar^2/m\bar{r}^2$

Coupling: $D = mp^2/\hbar^2\bar{r} \;\propto n^{1/2}$

Correspondence between classical (high temperature) and quantum (low temperature) domains ($\Gamma_D \;\Leftrightarrow\; D$) when $\Gamma(\Gamma_D) = \Gamma(D)$

# Gas-solid quantum phase transition



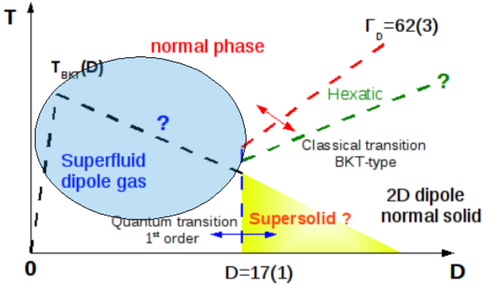

$D = 18$
**(solid)**

$D = 15$ **(gas)**
**strong coupling**

$D = 1$ **(gas)**
**weak coupling**

# Bosonic dipoles: Phase diagram

Thermodynamic phases as a function of $[T, D]$




**Phase transitions:**

- Crystallization at $T = 0$ [1]: $D = 17 \pm 1$
- Classical gas-solid transition [2]: $\Gamma_D \geq 62 \pm 3$
- Normal-superfluid transition, $T_{BKT}$

[1] Astrakharchik *et al.*, Phys. Rev. Lett. **98**, 060405 (2007); Büchler *et al.*, Phys. Rev. Lett. **98**, 060404 (2007); Mora *et al.*, Phys. Rev. B **76**, 064511 (2007)

[2] Kalia and Vashishta, J. Phys. C: Solid State Phys. **14**, 643 (1981)

# Collective modes analysis of 2D dipoles

**In-plane longitudinal mode**:

- *Random-phase-approximation* (RPA) [1]

$$\omega(q \to 0) \propto q^{3/2}, \quad \omega(q \to 0) \propto p^{1/2} \ (p - \text{dipole moment})$$

RPA limit does not exist due to divergence of the average Hartree field:

$$\langle U_{\rm D} \rangle_H = n \int d^2 r \, \frac{1}{r^3}$$

- DMC [2], Kachintsev [3], *Quasilocalized charge approximation* (QLCA) [4]

$$\omega(q \to 0) \propto q, \quad \omega(q \to 0) \propto p$$

RPA limit exist for the softened interaction: $U_{\rm sD} = U_{\rm D}(1 - e^{-r^2/d^2})$.

**Goal**: First principle results for $\omega(q \to 0)$ of bosonic dipoles for different coupling strengths $D \propto p^2$.

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[1] Joglekar, Balatsky, and Das Sarma *et al.*, Phys. Rev. B. **74**, 233302 (2006)

[2] Astrakharchik *et al.*, Phys. Rev. Lett. **98**, 060405 (2007)

[3] Kachintsev and Ulloa, Phys. Rev. B. **50**, 8715 (1994)

[4] Kalman, Hartmann, Donko and Golden, Phys. Rev. Lett. **98**, 236801 (2007)

# Outline

# Thermodynamic properties

- Equation of state ($T = 0$). Finite-size effects.

$$\epsilon(n) = \frac{E(n)}{N} = \epsilon_{\rm pot}(n) + \epsilon_{\rm kin}(n) \approx a_1\, n^{3/2} + a_2\, n^{5/4},$$

$$\epsilon_N(n) = \epsilon(n) - f_\epsilon(N),$$

$$f_\epsilon(N) = \frac{1}{2} \int_{L/2}^{\infty} \frac{p^2}{r^3}\, g(r)\, 2\pi r \mathrm{d}r + \tilde{f}_\epsilon(N)$$

- Chemical potential $\mu$. Inverse compressibility $\zeta$. Sound speed $c_T$.

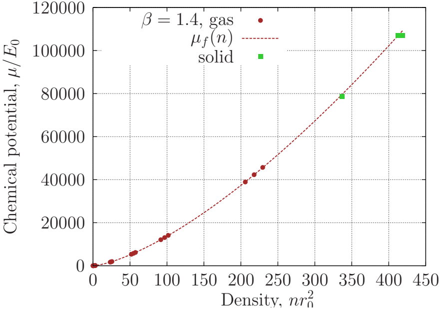
$$\mu(n) = \frac{\partial E(n)}{\partial N}, \quad \frac{1}{\kappa(n)} = \zeta(n) = \frac{\partial \mu(n)}{\partial n}, \quad m c_T^2(n) = n\, \zeta(n)$$

# PIMC simulations in the grand canonical ensemble

$$Z_{GCE} = \sum_{N=0}^{\infty} e^{\int_0^\beta \mu\, N(\tau)\, d\tau} Z_{CE}(N, V, \beta)$$

Worm algorithm: M.Boninsegni, N.Prokof'ev, and B.Svistunov, Phys. Rev. E **74, 036701 (2006)**

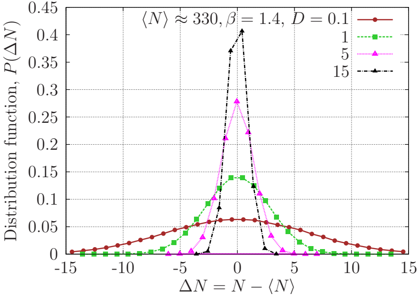
- Density $\langle n\rangle = \langle N\rangle_\mu / V$ vs $\mu$.




$$\mu(n) = 2.5\, a_1\, n^{3/2} + 2.25\, a_2\, n^{5/4}$$

- Compressibility: $\kappa VT = \langle (N - \langle N\rangle)^2\rangle_\mu$
- Probability of particle number fluctuation

**gas**



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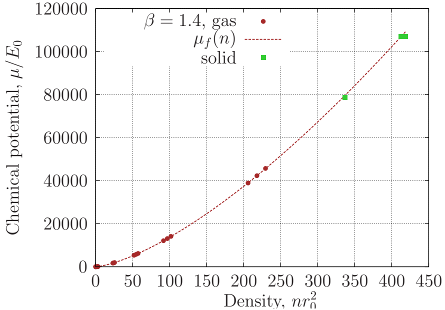
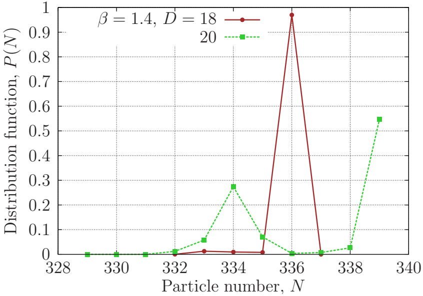



$$\mu(n) = 2.5\, a_1\, n^{3/2} + 2.25\, a_2\, n^{5/4}$$

- Compressibility: $\kappa V T = \langle (N - \langle N \rangle)^2 \rangle_\mu$
- Probability of particle number fluctuation

**solid**

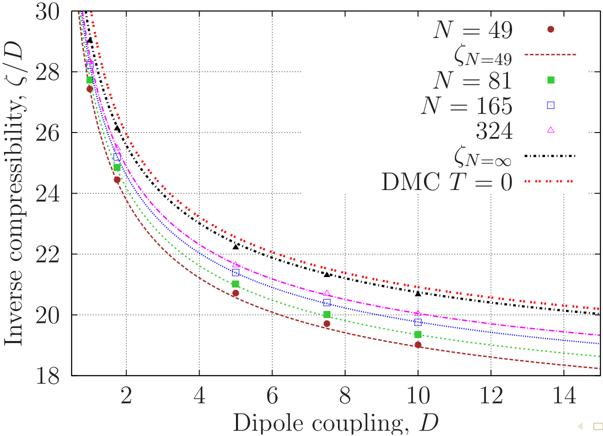


# Inverse compressibility: finite-size scaling

$$\frac{\zeta(n)}{m/\hbar^2} = \tilde{\zeta}_N(\tilde{n}) = \tilde{\zeta}(\tilde{n}) - \frac{2\pi\tilde{n}^{1/2}}{\sqrt{N/2}} = \frac{15}{4}\, a_1 \tilde{n}^{1/2} + \frac{45}{16}\, a_2\, \tilde{n}^{1/4} - \frac{2\pi\tilde{n}^{1/2}}{\sqrt{N/2}}$$

Fit results: $a_1 = 4.4906 \pm 0.015$, $a_2 = 4.3914 \pm 0.041$, $\tilde{n} = D^2$.

Temperature: $T \approx T_c/2$, $T_c = \pi\hbar^2 n_s(T)/2m \le \pi\hbar^2 n/2m = \pi/2$

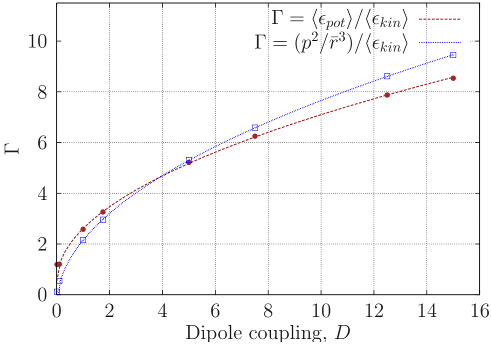



DMC: G. Astrakharchik *et al.*, Phys. Rev. Lett. **98**, 060405 (2007)

# Outline

# Classical $\Leftrightarrow$ Quantum mapping

- Correspondence $\Gamma_D \Leftrightarrow D$: $\epsilon_{\text{pot}}/\epsilon_{\text{kin}} = \Gamma(\Gamma_D) = \Gamma(D)$




**Quantum system**

$$\epsilon_{\text{pot}}(n) = a_1\, n^{3/2} + a_2\, n^{5/4},$$

$$\epsilon_{\text{kin}}(n) = b_1\, n^{5/4},$$

$$\Gamma = \frac{\epsilon_{\text{pot}}(n)}{\epsilon_{\text{kin}}(n)} = 2.076\, D^{1/2} + 0.531(4)$$

**Classical system**

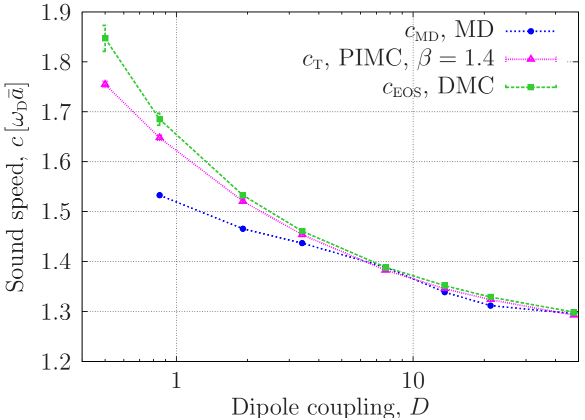
$$\Gamma = \frac{\epsilon_{\text{pot}}(\Gamma_D)}{\epsilon_{\text{kin}}(\Gamma_D)} = 0.793\,\Gamma_D + 0.579(4)$$

**Mapping relation**: $\Gamma_D \approx 2.62\sqrt{D}$

# Outline

# Sound speed vs dipole coupling: MD vs PIMC vs EOS

- Sound speed $[\omega_D \bar{a}]$ vs. coupling $D$ and $\Gamma_D = 2.62\sqrt{D}$: MD[1] ($c_{MD}$), DMC[2] ($c_{EOS}$) and PIMC ($c_T(n,\beta) = [n\zeta(n,\beta)/m]^{1/2}$). Dipole frequency $\omega_D^2 = 2\pi p^2 n/(m\bar{a}^3)$. (PIMC: $\beta = 1.4$. $\tilde{n} = D^2$).




Condensate fraction:
$\gamma(D) = n_0(D)/n$

$\gamma(15) \approx 1\%$,
$\gamma(5) \approx 11\%$,
$\gamma(1.75) \approx 23\%$,
$\gamma(0.1) \approx 66\%$

[1] K. Golden *et al.*, Phys. Rev. B **78**, 045304 (2008); [2] G. Astrakharchik *et al.*, Phys. Rev. Lett. **98**, 060405 (2007)

# Outline

# Feynman dispersion relation

- Gives an upper bound to the excitation spectrum[1]
  - $T = 0$: $\omega_F(q) = \hbar q^2/2mS(q, T = 0)$

    – The excited-state wavefunction

$$|\Psi_F\rangle = \sum_{i=1}^{N} f(\mathbf{r}_i)\,|\Psi_0\rangle$$

– Result of the variational calculations ($T = 0$): creation of a single density fluctuation of wavevector $q$

$$f(\mathbf{r}_i) = e^{i\mathbf{q}\mathbf{r_i}} \quad \Rightarrow \quad |\Psi_F\rangle = \hat{\rho}^+|\Psi_0\rangle,\ \hat{\rho}^+ = \sum_{i=1}^{N} e^{i\mathbf{q}\mathbf{r_i}}$$

– Energy of this state: $\omega_F(q) = \epsilon_q/S(q)$, $\epsilon_q = q^2/2m$

– Single-quasiparticle modes of infinite lifetime

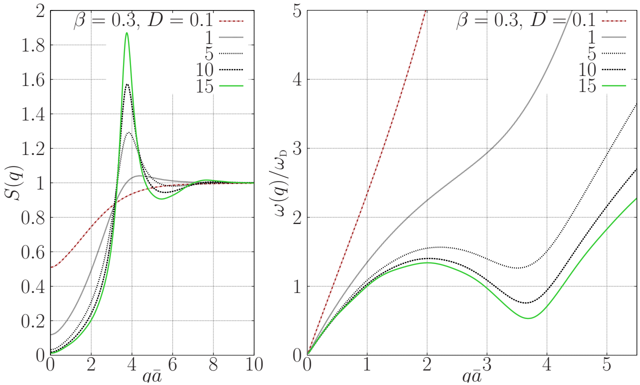
$$S(q,\omega) = S(q)\,\delta(\omega - \omega_F(q))$$

[1] A.Bijl, Physica **7**, 869(1940); R.Feynman, Phys.Rev. **94**, 262(1954)

# Feynman dispersion relation

- Gives an upper bound to the excitation spectrum[1]
  - $T = 0$: $\omega_F(q) = \hbar q^2/2mS(q, T = 0)$
  - $T \neq 0$: $\omega(q)\,\tanh\left[\beta\hbar\omega(q)/2\right] = \hbar q^2/2mS(q, \beta)$

**2D bosonic dipoles: static structure factor and Feynman dispersion**




[1] A.Bijl, Physica **7**, 869(1940); R.Feynman, Phys.Rev. **94**, 262(1954)

# Feynman dispersion relation

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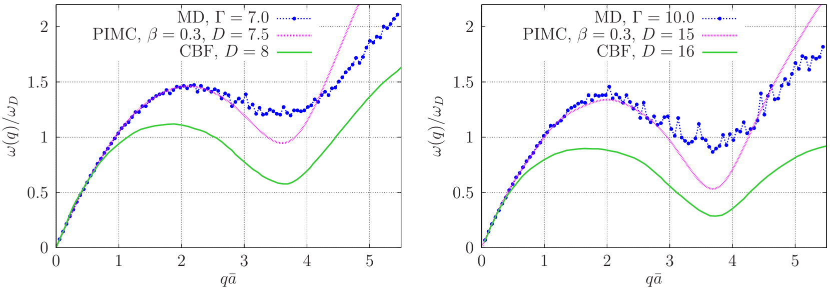
  - $T \neq 0$: $\omega(q) \tanh\left[\beta\hbar\omega(q)/2\right] = \hbar q^2/2mS(q, \beta)$
  - Generalization of the variational ansatz (Feynman,Cohen, Miller, Pines, Nozieres, Feenberg, ...)

**Correlated-basis-function approach (CBF)**

$$S(q,\omega) = -\frac{1}{\pi} Im \frac{S(q)}{\hbar\omega - \hbar\omega_F(q) - \sum(q,\omega) + i\nu}$$

# Feynman (PIMC) vs MD vs CBF




- $\omega(q)$ in the low-$q$ region is reproduced in all methods.
- Position of the roton minimum corresponds to maximum in $S(q) \Rightarrow$ correlation origin.
- Roton minimum is present also in classical MD but has a different depth.

**CBF**: F.Mazzanti et al., Phys.Rev.Lett **102**, 110405(2009)

# QLCA dispersion

- Density response function[1]

$$\chi(\mathbf{q},\omega) = \frac{nq^2/m\omega^2}{1-\Psi(q)\, nq^2/m\omega^2},$$

$$\Psi(q) = \frac{3\pi p^2}{q^2} \int_0^\infty \mathrm{d}r \frac{1}{r^4} g(r) \left[3 - 3J_0(qr) + 5J_2(qr)\right]$$

- QLCA dispersion relation

$$1-\Psi(q)\, nq^2/m\omega^2 = 0 \;\Rightarrow\; \omega(q) = q\sqrt{\frac{n}{m}\Psi(q)}$$

- Further improvements: Lindhard function, $nq^2/m\omega^2 \Rightarrow \chi^L(\mathbf{q},\omega)$

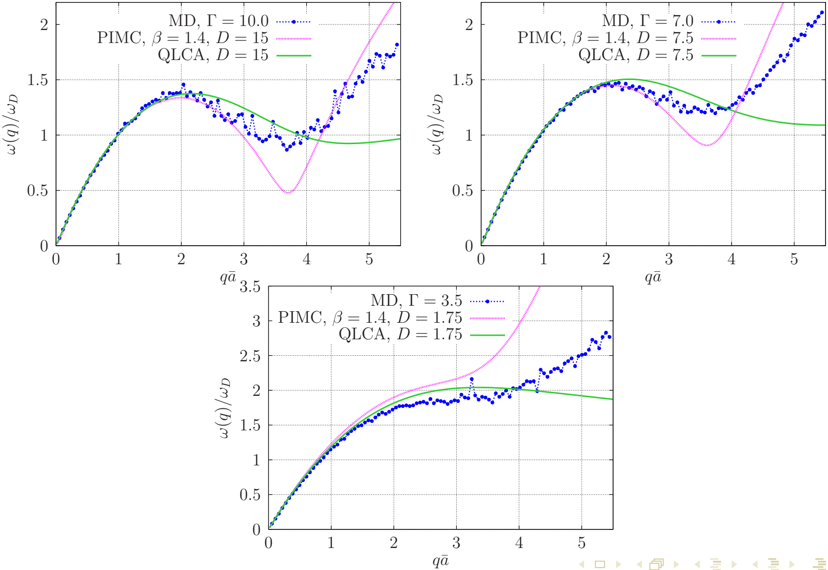
$$\chi^L(\mathbf{q},\omega) = \frac{1}{V}\sum_{\mathbf{k}} \frac{n_{\mathbf{k}-\mathbf{q}/2} - n_{\mathbf{k}+\mathbf{q}/2}}{\hbar\omega - (\hbar^2/m)\mathbf{k}\cdot\mathbf{q}}$$

– account degeneracy effects via the momentum distribution $n_{\mathbf{k}}$ $(N = \sum_{\mathbf{k}} n_{\mathbf{k}})$.

– strong coupling limit: $n_{\mathbf{k}} \sim e^{-\beta\epsilon_{\mathbf{k}}}$, $\epsilon_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m}$, $\chi^L(\mathbf{q}\to 0,\omega) \approx \frac{nq^2}{m\omega^2}$

[1] K. Golden *et al.*, Phys. Rev. B **78**, 045304 (2008)

# QLCA vs Feynman vs MD



# QLCA vs Feynman vs Compressibility

Table: 2D dipole superfluid gas ($\beta = 1.4$). Sound speed $[\omega_D \bar{a}]$ vs. coupling $D$.

| $D$ | $n_0/n$ | $c_{\mathrm{QLCA}}$, $N$=324 | $c_{\mathrm{F}}$, $N$=324 | $c_\kappa$, $N$=324 | $c_\kappa$, $N=\infty$ |
|---|---|---|---|---|---|
| 0.1 | $\sim 66\%$ | 2.064(2) | 2.2027(15) | – | – |
| 1.75 | $\sim 23\%$ | 1.480(3) | 1.512(4) | 1.514(3) | 1.533(4) |
| 5.0 | $\sim 11\%$ | 1.393(3) | 1.401(2) | 1.395(3) | 1.417(3) |
| 7.5 | $\sim 5\%$ | 1.369(3) | 1.370(2) | 1.362(3) | 1.385(3) |
| 10 | $\sim 2\%$ | 1.354(4) | 1.352(3) | 1.342(3) | 1.365(3) |
| 15 | $\sim 1\%$ | 1.335(3) | 1.3305(40) | 1.317(3) | 1.341(3) |

- Discrepancy between $c_{\mathrm{QLCA}}$ and $c_\kappa, c_F$ increases of the condensate fraction $n_0(D)/n$.

# Outline

# Summary

- First principle results (PIMC) on the density dependence of $\zeta(n)$ and $c_T(n)$
- We used PIMC+Feynman, MD and QLCA to determine the mode dispersion $\omega(q)$
- Up to $q \leq 2$ all methods reproduce very similar behaviour of $\omega(q)$
- At strong and equal coupling there is no difference between classical and quantum longitudinal modes in 2D dipole systems
- We coincide with the variational solution (CBF) for the low-$q$ region
- Both QLCA and Feynman are not succesfull in predicting temperature dependence of the dispersion if we go through the BKT transition

**Outlook**

- More accurate theories for $\omega(q)$: QLCA+ Lindhard, dynamical structure factor (analytical continuation from imaginary time), ...

Thank you for your attention!

# Outline

# Thermodynamic properties

- Equation of state ($T = 0$)

$$\epsilon(n) = \frac{E(n)}{N} = \epsilon_{\text{pot}}(n) + \epsilon_{\text{kin}}(n) = a_1\, n^{3/2} + a_2\, n^{5/4}$$

- Chemical potential. Compressibility

$$\mu(n) = \frac{\partial E(n)}{\partial N}, \quad \zeta(n) = \frac{1}{\kappa(n)} = \frac{\partial \mu(n)}{\partial n} = \frac{15}{4}\, a_1 n^{1/2} + \frac{45}{16}\, a_2\, n^{1/4}$$

- Finite size correction

$$\epsilon(n) = \epsilon_N(n) + \epsilon_{\text{corr}}(n) + f_\epsilon(1/N), \quad \zeta(n) = \zeta_N(n) + \zeta_{\text{corr}}(n) + f_\zeta(1/N),$$

$$\epsilon_{\text{corr}}(n) = \frac{1}{2} \int_{L/2}^{\infty} \frac{p^2}{r^3}\, g(r)\, 2\pi r \mathrm{d} r \approx \frac{\pi p^2 n}{L/2}, \quad g(r > L/2) = n,\ L = N^{1/2},$$

$$\zeta_{\text{corr}}(n) = \frac{1}{V} \frac{\partial^2}{\partial n^2} \left(N \cdot \epsilon_{\text{corr}}(n)\right) \approx \frac{2\pi p^2}{L/2}$$

System of units: $r_0 = \frac{mp^2}{\hbar^2}$, $E_0 = \frac{\hbar^2}{mr_0^2}$ $\Rightarrow$

$\tilde{n} = nr_0^2,\ \tilde{\epsilon}(n) = \epsilon(n)/E_0,\ \tilde{\zeta} = \zeta/(m/\hbar^2)$

# Approaches to collective mode dispersion

- Acoustic sound speed via compressibility $\kappa(n,\beta)$ (Grand Canonical bosonic PIMC)

$$\omega(q \to 0) = c_T\, q, \quad c_T(n,\beta) = [n/m\kappa(n,\beta)]^{1/2}$$

- Sound speed $c_{\mathrm{MD}}(\Gamma_D)$ vs $c_T(D,\beta \gg 1)$ using correspondence $\Gamma_D \Leftrightarrow D$ between strongly coupled classical and quantum system
- Feynman excitation spectrum (upper bound), $\omega(q) \leq \omega_F(q)$
- Dispersion relation $\omega_{\mathrm{MD}}(q)$ vs $\omega_F(q)$ using correspondence $\Gamma_D \Leftrightarrow D$
- QLCA dispersion $\omega_{\mathrm{QLCA}}(q)$ using as an input pair distribution function $g(r)$ from numerical simulations (MD, PIMC)

# Sound speed

Table: 2D dipole liquid: MD[1], DMC[2] ($c_{\mathrm{EOS}}$) and PIMC ($c_T(\beta)$), sound speed $[\omega_D \bar{a}]$ vs. coupling ($\Gamma_D, D$). Dipole frequency $\omega_D^2 = 2\pi p^2 n/(m\bar{a}^3)$. $\beta = 1.4$

| $\Gamma_D$ | $c_{\mathrm{MD}}$ | $D$ | $c_T$ | $c_{\mathrm{EOS}}(T=0)$ |
|---|---|---|---|---|
| **2** | 1.533 | **0.85** | 1.648(5) | 1.685(12) |
| **3** | 1.466 | **1.91** | 1.521(4) | 1.533(7) |
| **4** | 1.437 | **3.40** | 1.454(4) | 1.461(4) |
| **6** | 1.389 | **7.65** | 1.383(3) | 1.389(3) |
| **8** | 1.339 | **13.61** | 1.346(3) | 1.352(2) |
| **10** | 1.312 | **21.26** | 1.324(3) | 1.329(2) |
| **15** | 1.296 | **47.8** | 1.293(3) | 1.299(2) |
| **20** | 1.276 | **85** | 1.277(3) | 1.2835(10) |
| **30** | 1.246 | **181** | 1.262(3) | 1.2678(10) |
| lattice (>**62**) | 1.283 | | – | – |

[1] K. Golden *et al.*, Phys. Rev. B **78**, 045304 (2008)

[2] G. Astrakharchik *et al.*, Phys. Rev. Lett. **98**, 060405 (2007)

# General many-body problem

- System Hamiltonian:

$$\hat{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\boldsymbol{\nabla}_i^2 + \sum_{i=1}^{N} V_{\text{ext}}(\mathbf{r}_i) + \sum_{i<j} V_{int}(\mathbf{r}_i, \mathbf{r}_j)$$

- N-particle density operator [ $NVT$-ensemble], $\hat{H}|j\rangle = E_j|j\rangle$

$$\hat{\rho} = \frac{1}{Z}\sum_j e^{-\beta E_j}\,|j\rangle\,\langle j| = e^{-\beta\hat{H}}, \quad \beta = 1/k_B T$$

- Thermodynamic averages

$$\langle A\rangle = \frac{\sum_j e^{-\beta E_j}\langle j|\hat{A}|j\rangle}{\sum_j e^{-\beta E_j}\langle j|j\rangle} = \frac{\text{Tr}\left[\hat{\rho}\hat{A}\right]}{\text{Tr}\left[\hat{\rho}\right]}.$$

- One-particle (reduced) density matrix in the coordinate presentation, i.e. $|j\rangle = |\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\rangle$

$$n(\mathbf{r}, \mathbf{r}') = \int \mathrm{d}\mathbf{r}_2 \ldots \mathrm{d}\mathbf{r}_N\,\langle \mathbf{r}, \mathbf{r}_2, \ldots, \mathbf{r}_N|\hat{\rho}|\mathbf{r}', \mathbf{r}_2, \ldots, \mathbf{r}_N\rangle$$

# Density matrix and path integrals

- Density matrix has to be symmetric (bosons) or antisymmetric (fermions): $\hat{\rho} \longrightarrow \hat{\rho}^{S/A}$
- Construct new states by permutation operator $\hat{P}$ and symmetrize

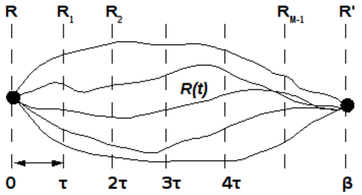
$$\langle R|\hat{\rho}^{S/A}|R'\rangle = \rho^{S/A}(R, R'; \beta) = \frac{1}{N!}\sum_{P=1}^{N!}(\pm 1)^{\delta P}\rho(R, \hat{P}R'; \beta)$$

[$R = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$ is the $N$-particle radius vector.]

- **Feynman's idea:** evaluate $\rho^{S/A}$ via path integrals

*product property:* $\hat{\rho}(\beta) = e^{-\beta\hat{H}} = \left[e^{-\frac{\beta}{M}\hat{H}}\right]^M = [\hat{\rho}(\tau)]^M$ , $\tau = \frac{\beta}{M} = \frac{1}{Mk_B T}$

$$\rho(R, \hat{P}R'; \beta) = \int \mathrm{d}R_1 \ldots \mathrm{d}R_{M-1}\, \rho(R, R_1; \tau)\rho(R_1, R_2; \tau)\cdots\rho(R_{M-1}, \hat{P}R'; \tau)$$



# Definition of BEC

**Definition of BEC for interacting systems**[1]

– *One-particle* density matrix ($T = 0$)

$$n(\mathbf{r},\mathbf{r}') = \int \ldots \int d\mathbf{r}_2 \ldots d\mathbf{r}_N \Psi^*(\mathbf{r},\mathbf{r}_2,\ldots,\mathbf{r}_N)\Psi(\mathbf{r}',\mathbf{r}_2,\ldots,\mathbf{r}_N) = \langle \hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r}')\rangle$$

– Diagonalization in a complete orthogonal basis set:

$$n(\mathbf{r},\mathbf{r}') = n_0\chi_0^*(\mathbf{r})\chi_0(\mathbf{r}') + \sum n_i\chi_i^*(\mathbf{r})\chi_\mathbf{i}(\mathbf{r}').$$

– Definition of BEC: there exist at least one eigenstate $\chi_0$ with $n_0/N \lesssim 1$

**BEC $\equiv$ existence of the off-diagonal long-range order (ODLRO)**

$$n(\mathbf{r},\mathbf{r}')|_{|\mathbf{r}-\mathbf{r}'|\to\infty} \propto \begin{cases} 3D: \ n_0, & \text{for } T < T_c \\ 2D: \ |\mathbf{r}-\mathbf{r}'|^{-\nu}, \ \nu = mk_BT/[2\pi\hbar^2\rho_s(T)] & \text{for } T < T_c \\ e^{-|\mathbf{r}-\mathbf{r}'|/\mathbf{a}}, & \text{for } T > T_c \text{ (no ODLRO)} \end{cases}$$

**ODLRO $\Rightarrow$ NCRI**[2]

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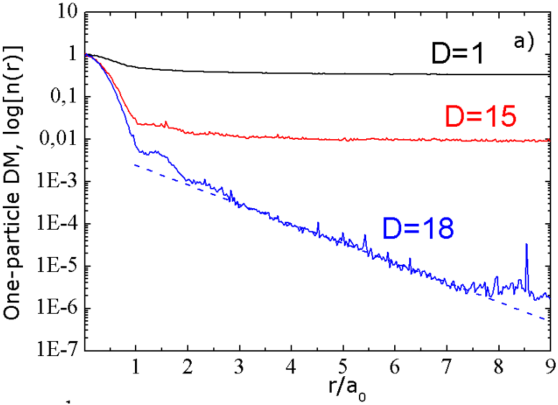
[1] O. Penrose, L. Onsager, Phys. Rev. **104**, 5767 (1956); C.N. Yang, Rev. Mod. Phys. **34**, 694 (1962)

[2] Yu Shi, Phys. Rev. B 72, 014533 (2005)

# Off-diagonal long-range order

$D \geq 18$: $n(\mathbf{r}, \mathbf{r}') \propto e^{-a|\mathbf{r}-\mathbf{r}'|}$ *(normal phase)*

$D \leq 15$: $n(\mathbf{r}, \mathbf{r}') \propto |\mathbf{r} - \mathbf{r}'|^{-\nu}$ *(superfluid phase)*, $\nu = \frac{mk_B T}{2\pi\hbar^2 \rho_s(T)}$




$n_0 \approx 30\%$

$\rho_s/\rho \approx 0.92$

$n_0 \approx 1\%$

$\rho_s/\rho \lesssim 10^{-5}$