Velocity distributions and kinetic equations for plasmas including Levy-type noise

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Our task

• Theoreticians like Gaussian distributions and in particular the Maxwell distribution of velocities which was found already in 1866.
• In nature we find sometimes weak deviations (noneq gases and plasmas) and sometimes big deviations from Maxwell (high energy events, explosions of clusters or heavy nuclei, shocks, TOKAMAKs, laser-radiated D-clusters,.....)
• problem of hydrodynamics (E.Son)
Open questions

• Sometimes we have no good theory but assume hypothetically strong deviations as
  • gas discharges (Dryuvestein distributions)
  • anomalous diffusion at the edge of Tokamak devices
  • flow problems, ...........
• The high neutron emission from irradiated clusters as due to high energy wings???
What is known about Coulomb clusters exc by fs laser beams


• \(10^{\{-14\}}\) --- \(10^{\{-13\}}\) ---- \(10^{\{-12\}}\) ---- \(10^{\{-11\}}\)----

• laser pulse  cluster life time  formation of uniform plasma

• Coulomb energy plays a large role \(R(t) \sim c t\) (\(c\sim\)sound speed)

• Note: our estimate of \(R(t)\) is more complicated but leads also to a quasilinear dependence
• Laser pulse filament
• diam. 200 μm, length 1 mm
• average ion density:
  • $2 \times 10^{19} / \text{cm}^3$
• peak average ion energy ~
  • 12 keV
• Total neutron yield per shot ~
  5000-20000
Outline of this talk

• We explain the difference between Gauss and Levy distributions
• we study convolutions of Gauss and Levy distributions,
• we solve the Langevin equation with pure Levy and mixed noise sources,
• we discuss the high energy tails and possible applications
What is a Levy distribution?

\[ W_L(\beta, \alpha) = \int_0^\infty \cos(\beta t) \exp(-t^\alpha) dt, \]

\[ W_L(\beta, \alpha) \sim 1/\beta^{\alpha+1} \quad \text{if} \quad \alpha < 2 \]

\[ W_L(\beta, 2) \sim \exp(-c\beta^2) \quad \text{Gauss distr} \]

\[ W_L(\beta, 1) \sim \frac{a}{b + \beta^2} \quad \text{Cauchy distr} \]

- Levy distributions are quite general distr which contain Gauss- and Cauchy- distributions as special cases. Specific property = long tails
- Note: in 3d we have a (t \sin t) instead of (\cos t)
Levy distr play a role in plasma physics since 1919 when Holtsmark showed that the 3d-microfield distributions are Levy-type (index 1.5)

\[
W_{MF}(\beta, \alpha) = C \int_{0}^{\infty} t \sin(\beta t) \exp(-t^{\alpha}) dt,
\]

\[
W_{MF}(\beta, \alpha) \sim 1/\beta^{\alpha+1}, \ldots E_H = 1.2en^{2/3}
\]

- Here beta = E / E_n
- E_n charactristic field (E_H - Holtsmark field)
- alpha = 1.5 for Holtsmark distributions
- Mayer/Broyles fould alpha =2 for dense plasmas
- MRom found alpha = 1 for Kepler scattering
- Our simul: index changes in range 0.5<alpha<1.5,
Levy exponents of the tails from simulations (Sadykova/Valuev workshop 2009)

- In the present example Gamma=2 we find
- $\alpha < 1$ for H
- $\alpha \sim 1.5$ for neutrals
- $\alpha \sim 2$ for alkali
- Microfield distributions are appr of Levy-type but $1 < \alpha < 2$ and $E_n$ are free parameters, depending on species, $n$, $T$, the main body is Gaussian
Here we are interested not in Microfield distributions but in Velocity Distributions which are generated by microscopic stochastic forces: Gaussian collisions + Holtsm-electr microfields.

- Collisions corr to Gauss-distributed forces, microfields to Levy-distributed forces. In a first appr we obtain the integral distribution by a convolution of Gaussian with Levy-distributions

\[
W(y) = \int dz W_G(y - z) W_L(z)
\]

\[
W(y) = \int_{-\infty}^{\infty} dk \cos(ky) \exp(-at^\alpha - bt^2)
\]

\[
W(y) \sim \frac{1}{y^{\alpha+1}}
\]
Comparison of Gauss- with Cauchy distr and convoluted Gauss-Cauchy distr
For Gaussian stochastic forces, follows the standard Fokker-Planck equation:

\[
\frac{\partial}{\partial t} f(v, t) = \nabla [\gamma vf(v, t)] + D \nabla^2 f(v, t)
\]

Maxwell solutions are given by:

\[
f_0(v) = C \exp \left[ -\frac{\gamma v^2}{2D} \right] = C \exp \left[ -\frac{mv^2}{2k_B T} \right]
\]
For Levy forces follows a kinetic equation with fractal derivatives

\[ \frac{\partial}{\partial t} f(v, t) = \nabla[\gamma v f(v, t)] + D \nabla^\alpha f(v, t) \]

\[ \frac{\partial}{\partial t} f(k, t) = -\gamma k \frac{\partial}{\partial k} f(k, t) + D k^\alpha f(k, t) \]
The velocity distribution function for Levy noise

\[ \frac{\partial}{\partial t} f(k, t) = -\gamma_0 k \frac{\partial}{\partial k} f(k, t) + D k^\alpha f(k, t) \]

\[ f_0(k) = \exp \left[ -D(t) |k|^\alpha \right]; D(t) = \frac{qE_n}{m\alpha \gamma_0} [1 - \exp(-\alpha \gamma_0 t)] \]

\[ W(|v|) = \frac{2|v|}{\pi} \int_0^\infty t \sin(|v| t) \exp(-t^\alpha) dt \sim \frac{d(t)}{\alpha |v|^\alpha+1} \]

\[ \alpha = 1 \longrightarrow W(|v|) = \frac{4d(t) |v|^2}{\pi[d(t)+|v|^2]^2} \]
The velocity distribution for Gauss+Levy forces

\[ \frac{\partial}{\partial t} f(k,t) = -\gamma_0 k \frac{\partial}{\partial k} f(k,t) + (D_\alpha k^\alpha + D_2 k^2) f(k,t) \]

\[ f_0(k) = \exp \left[ -d_\alpha(t) D_\alpha |k|^\alpha - d_2(t) D_2 k^2 \right] \]

\[ d_\alpha(t) = \frac{1}{\alpha \gamma} [1 - \exp(-\alpha \gamma t)] \]

\[ W(|v|) = \frac{2|v|}{\pi} \int_0^\infty k v \sin(k |v|) \exp(-\frac{D_\alpha |k|^\alpha}{\alpha \gamma} - \frac{D_2 k^2}{2\gamma}) dt \sim \frac{d(t)}{\alpha |v|^{\alpha+1}} \]
Distr of the modulus of the velocity (root of kinetic energy)
The energy distribution for Gauss+Levy forces

\[ W(\varepsilon) = C \int_{0}^{\infty} k \sqrt{2m\varepsilon} \sin(k \sqrt{2m\varepsilon}) \exp\left(-\frac{D_\alpha k^\alpha}{\alpha \gamma} - \frac{D_2 k^2}{2\gamma}\right) dk \]

\[ W(\varepsilon) \sim \frac{\text{const}}{\alpha \varepsilon^{\alpha + 1/2}}, \quad W(\varepsilon > \varepsilon_0) \sim \frac{\text{const}}{\alpha (\alpha + 1) \varepsilon^{\alpha - 1/2}} \]
Difference between Gauss and Cauchy distr is for \((E/kBT \sim 10)\) already about 140, increases quickly
The problem of nuclear fusion (Gamov model)

- Fusion occurs as the result of tunneling of protons/deuterons through the Coulombic barrier: Enhancement requires influencing the barrier during the reaction time.
- or changing velocity distribution.
Take Thompson cross section of fusion average over energy distr (Cauchy)

\[ \sigma(U) = \frac{A}{U} \exp\left(-\frac{B}{\sqrt{U}}\right) \]

\[ \frac{dN^n}{dt} = Nn(t) \nu \sigma(u) \]

\[ \frac{dN^n}{dt} = Nn(t) \frac{2Ad(t)}{\pi B^2} \int_0^\infty du \exp\left[-\frac{B}{\sqrt{u}}\right] \frac{B^2}{\left[u + d(t)(m/2)\right]^2} \]
The Thompson semi-empirical
REACTION CROSS-SECTION
left: Thompson formula;    right: data for several reactions DD=red
LLNL EXPERIMENTS
(PRL 85, 3640 (2000)) vs. our estimate

Exp: Shot about 50 ps

- average ion density:
  - $2 \times 10^9$ cm$^{-3}$

- peak average ion energy:
  - $\sim 12$ keV

Total neutron yield
- 100-1000 neutrons/shot
- estimate $\sim 30-100$
Interpretation: Fast ionization of clusters by strong laser pulses, explosion due to Coulomb repulsion

- In subsequent collisions sometimes very high fields are created which accelerate D-D fusion
- LLNL-group: main role play “collisions of clusters”  
  - Our estimate: coll times too short cross sections too small. The observation of neutrons 100 - 1000 neutrons / ps remains unexplained
Our interpretation

• The strong repulsive forces in Coulomb clusters lead to deviations from Maxwell distributions with power law decay.
• Short time high energy wings appear in the range $10 \text{ fs} - 100 \text{ fs}$ decay within 1 ps.
• High energetic protons/deuterons may penetrate the fusion barrier
• Theoretical approach: study the velocity distributions = Levy distributions with long tails.
• enhancement of fusion rates due to the long tails in the velocity distribution
   • On Levy processes:
   • W. Ebeling, M.Yu. Romanovsky, Microfields and Fusion,
   *Contrib. Plasma Phys.* **49** 195 (2009),